Chapter 5 Probability

Section 5.1 Probability Rules

Objectives

1. Apply the rules of probabilities
2. Compute and interpret probabilities using the empirical method
3. Compute and interpret probabilities using the classical method
4. Use simulation to obtain data based on probabilities
5. Recognize and interpret subjective probabilities

Probability is a measure of the likelihood of a random phenomenon or chance behavior occurring. Probability describes the long-term proportion with which a certain **outcome** will occur in situations with short-term uncertainty.

Activity Coin Flips

Go to www.pearsonhighered.com/sullivanstats and select the Dice Rolling applet. Or, create an applet in StatCrunch by selecting Applets > Simulation > Dice Rolling. Let the number of sides by 10 and the number of dice be 1. Click Compute!. Under Event in the applet notice, "For the Sum of 1 rolls". Use the pull-down menu and select = and enter 4 in the dialogue box. Click on Convergence.

(a) Click 5 runs four times. Notice the applet rolls a single die a total of 20 times. What proportion of the rolls resulted in a "four" being rolled?

(b) Click 5 runs four more times (for a total of 40 rolls). What proportion of the rolls resulted in a "four" being rolled?

(c) Click Reset. Click 1000 runs. What proportion of the rolls results in a "four" being rolled? What happens to the relative frequency with which a four is observed as the number of rolls increases?

(d) Click Reset. Click 1000 runs. What proportion of the rolls results in a "four" being rolled? What happens to the relative frequency with which a four is observed as the number of rolls increases? Is this the same result as you obtained in Part (c)? Is the behavior in the short run (fewer rolls) the same as it was in Part (c)?

(e) Based on the results of Parts (c) and (d), what would you conjecture is the probability of rolling a four with a single 10-sided die?

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Probability deals with experiments that yield random short-term results or **outcomes**, yet reveal long-term predictability.

The long-term proportion in which a certain outcome is observed is the probability of that outcome.

The Law of Large Numbers

As the number of repetitions of a probability experiment increases, the proportion with which a certain outcome is observed gets closer to the probability of the outcome.

In probability, an **experiment** is any process that can be repeated in which the results are uncertain.

The **sample space**, S, of a probability experiment is the collection of all possible outcomes.

An **event** is any collection of outcomes from a probability experiment. An event consists of one outcome or more than one outcome. We will denote events with one outcome, sometimes called *simple events*, e_i. In general, events are denoted using capital letters such as E.

Example 1 Identifying Events and the Sample Space of a Probability Experiment

(a) Identify the outcomes of the probability experiment.

(b) Determine the sample space.

(c) Define the event E = "have one boy".

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1 Apply the Rules of Probabilities

Rules of Probabilities

1. The probability of any event E, $P(E)$, must be greater than or equal to 0 and less than or equal to 1. That is, $0 \leq P(E) \leq 1$.
2. The sum of the probabilities of all outcomes must equal 1. That is, if the sample space $S = \{e_1, e_2, \ldots, e_n\}$, then

$$P(e_1) + P(e_2) + \cdots + P(e_n) = 1$$

A **probability model** lists the possible outcomes of a probability experiment and each outcome's probability. A probability model must satisfy rules 1 and 2 of the rules of probabilities.

Example 2 A Probability Model

In a bag of peanut M&M milk chocolate candies, the colors of the candies can be brown, yellow, red, blue, orange, or green. Suppose that a candy is randomly selected from a bag. The table shows each color and the probability of drawing that color. Verify this is a probability model.

Color	**Probability**
Brown	0.12
Yellow	0.15
Red	0.12
Blue	0.23
Orange	0.23
Green	0.15

■

If an event is **impossible**, the probability of the event is 0. If an event is a **certainty,** the probability of the event is 1.

An **unusual event** is an event that has a low probability of occurring.

2 Compute and Interpret Probabilities Using the Empirical Method

Approximating Probabilities Using the Empirical Approach

The probability of an event E is approximately the number of times event E is observed divided by the number of repetitions of the experiment.

$$P(E) \approx \text{relative frequency of } E = \frac{\text{frequency of } E}{\text{number of trials of experiment}} \quad \textbf{(1)}$$

Example 3 Using Relative Frequencies to Build a Probability Model

Pass the Pigs™ is a Milton-Bradley game in which pigs are used as dice. Points are earned based on the way the pig lands. There are six possible outcomes when one pig is tossed. A class of 52 students rolled pigs 3,939 times. The number of times each outcome occurred is recorded in the table.

Outcome	**Frequency**
Side with no dot	1344
Side with dot	1294
Razorback	767
Trotter	365
Snouter	137
Leaning Jowler	32

(a) Use the results of the experiment to build a probability model for the way the pig lands.

(b) Estimate the probability that a thrown pig lands on the "side with dot".

(c) Would it be unusual to throw a "Leaning Jowler"?

3 Compute and Interpret Probabilities Using the Classical Method

The classical method of computing probabilities requires **equally likely outcomes.**

An experiment is said to have **equally likely outcomes** when each simple event has the same probability of occurring.

Computing Probability Using the Classical Method

If an experiment has n equally likely outcomes and if the number of ways that an event E can occur is m, then the probability of E, $P(E)$, is

$$P(E) = \frac{\text{number of ways that } E \text{ can occur}}{\text{number of possible outcomes}} = \frac{m}{n} \qquad (2)$$

So, if S is the sample space of this experiment,

$$P(E) = \frac{N(E)}{N(S)} \qquad (3)$$

where $N(E)$ is the number of outcomes in E, and $N(S)$ is the number of outcomes in the sample space.

Example   Computing Probabilities Using the Classical Method

Suppose a “fun size” bag of M&Ms contains 9 brown candies, 6 yellow candies, 7 red candies, 4 orange candies, 2 blue candies, and 2 green candies. Suppose that a candy is randomly selected.

(a) What is the probability that it is yellow?

(b) What is the probability that it is blue?

(c) Comment on the likelihood of the candy being yellow versus blue.

4 Use Simulation to Obtain Data Based on Probabilities

The applet we used to generate outcomes from rolling a ten-sided die is an example of using simulation to obtain probabilities.

5 Recognize and Interpret Subjective Probabilities

A **subjective probability** of an outcome is a probability obtained on the basis of personal judgment.

For example, an economist predicting there is a 20% chance of recession next year would be a subjective probability.

EXAMPLE Empirical, Classical, or Subjective Probability

In his fall 1998 article in **Chance Magazine**, ("A Statistician Reads the Sports Pages," pp. 17-21,) Hal Stern investigated the probabilities that a particular horse will win a race. He reports that these probabilities are based on the amount of money bet on each horse. When a probability is given that a particular horse will win a race, is this empirical, classical, or subjective probability?

Subjective because it is based upon people's feelings about which horse will win the race. The probability is not based on a probability experiment or counting equally likely outcomes. ■

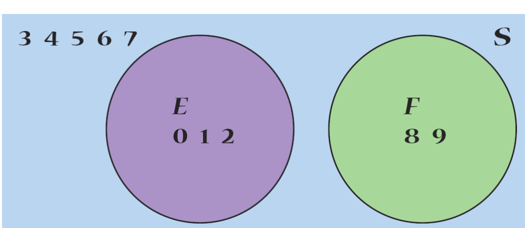
5.2 The Addition Rule and Complements

Objectives

1. Use the Addition Rule for Disjoint Events
2. Use the General Addition Rule
3. Compute the probability of an event using the Complement Rule

Two events are **disjoint** if they have no outcomes in common. Another name for disjoint events is **mutually exclusive** events.

We often draw pictures of events using **Venn diagrams.** These pictures represent events as circles enclosed in a rectangle. The rectangle represents the sample space, and each circle represents an event. For example, suppose we randomly select a chip from a bag where each chip in the bag is labeled 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Let E represent the event "choose a number less than or equal to 2," and let F represent the event "choose a number greater than or equal to 8." These events are disjoint as shown in the figure.



$$P(E)=\frac{N(E)}{N(S)}=\frac{3}{10}=0.3$$

$$P(F)=\frac{N(F)}{N(S)}=\frac{2}{10}=0.2$$

$$P(E \text{ or } F)=\frac{N(E \text{ or } F)}{N(S)}=\frac{5}{10}=0.5$$

$$=P(E)+P(F)=0.3+0.2=0.5$$

Addition Rule for Disjoint Events

If E and F are disjoint (or mutually exclusive) events, then

$$P(E \text{ or } F) = P(E) + P(F)$$

The Addition Rule for Disjoint Events can be extended to more than two disjoint events.

In general, if *E, F, G*, . . . each have no outcomes in common (they are pairwise disjoint), then

$$P(E \text{ or } F \text{ or } G \text{ or } \ldots) = P(E) + P(F) + P(G) \cdots$$

Example The Addition Rule for Disjoint Events

The probability model shows the distribution of the number of rooms in housing units in the United States.

Number of Rooms in Housing Unit	Probability
One	0.010
Two	0.032
Three	0.093
Four	0.176
Five	0.219
Six	0.189
Seven	0.122
Eight	0.079
Nine or more	0.080

(a) Verify that this is a probability model.

(b) What is the probability a randomly selected housing unit has two or three rooms?

(c) What is the probability a randomly selected housing unit has one or two or three rooms?

2 Use the General Addition Rule

Skip this objective

3 Compute the Probability of an Event Using the Complement Rule

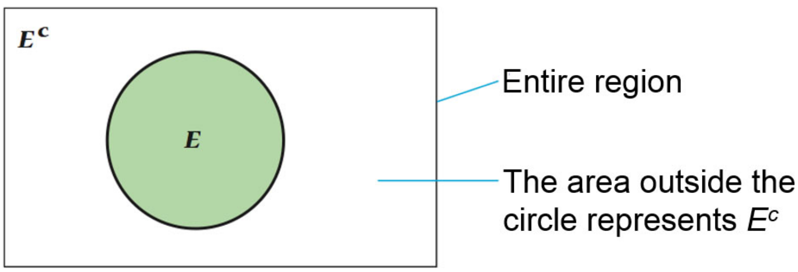
Complement of an Event

Let S denote the sample space of a probability experiment and let E denote an event. The **complement of** $\boldsymbol{E}$, denoted E^c, is all outcomes in the sample space S that are not outcomes in the event E.

Complement Rule

If E represents any event and E^c represents the complement of E, then

$$P(E^c) = 1 - P(E)$$



EXAMPLE Illustrating the Complement Rule

According to the American Veterinary Medical Association, 31.6% of American households own a dog. What is the probability that a randomly selected household does not own a dog?

Example Computing Probabilities Using Complements

The data to the right represent the travel time to work for residents of Hartford County, CT.

(a) What is the probability a randomly selected resident has a travel time of 90 or more minutes?

Travel Time	Frequency
Less than 5 minutes	24,358
5 to 9 minutes	39,112
10 to 14 minutes	62,124
15 to 19 minutes	72,854
20 to 24 minutes	74,386
25 to 29 minutes	30,099
30 to 34 minutes	45,043
35 to 39 minutes	11,169
40 to 44 minutes	8,045
45 to 59 minutes	15,650
60 to 89 minutes	5,451
90 or more minutes	4,895

Source: United States Census Bureau

(b) Compute the probability that a randomly selected resident of Hartford County, CT will have a commute time less than 90 minutes.

Section 5.3 Independence and the Multiplication Rule

Objectives

1. Identify independent events
2. Use the Multiplication Rule for Independent Events
3. Compute at-least probabilities

1 Identify Independent Events

Two events *E* and *F* are **independent** if the occurrence of event *E* in a probability experiment does not affect the probability of event *F*. Two events are **dependent** if the occurrence of event *E* in a probability experiment affects the probability of event *F*.

EXAMPLE Independent or Not?

(a) Suppose you draw a card from a standard 52-card deck of cards and then roll a die. The events "draw a heart" and "roll an even number" are independent because the results of choosing a card do not impact the results of the die toss.

(b) Suppose two 40-year old women who live in the United States are randomly selected. The events "woman 1 survives the year" and "woman 2 survives the year" are independent.

(c) Suppose two 40-year old women live in the same apartment complex. The events "woman 1 survives the year" and "woman 2 survives the year" are dependent.

Disjoint Events versus Independent Events Disjoint events and independent events are different concepts. Recall that two events are disjoint if they have no outcomes in common, that is, if knowing that one of the events occurs, we know the other event did not occur. Independence means that one event occurring does not affect the probability of the other event occurring. Therefore, knowing two events are disjoint means that the events are not independent.

2 Use the Multiplication Rule for Independent Events

Multiplication Rule for Independent Events

If E and F are independent events, then

$$P(E \text{ and } F) = P(E) \cdot P(F)$$

EXAMPLE Computing Probabilities of Independent Events

The probability that a randomly selected female aged 60 years old will survive the year is 99.186% according to the National Vital Statistics Report, Vol. 47, No. 28. What is the probability that two randomly selected 60 year old females will survive the year?

EXAMPLE Computing Probabilities of Independent Events

A manufacturer of exercise equipment knows that 10% of their products are defective. They also know that only 30% of their customers will actually use the equipment in the first year after it is purchased. If there is a one-year warranty on the equipment, what proportion of the customers will actually make a valid warranty claim?

Multiplication Rule for *n* Independent Events

If events $E_1, E_2, E_3, \ldots, E_n$ are independent, then

$$P(E_1 \text{ and } E_2 \text{ and } E_3 \text{ and } \ldots \text{ and } E_n) = P(E_1) \cdot P(E_2) \cdot \cdots \cdot P(E_n)$$

EXAMPLE Illustrating the Multiplication Principle for Independent Events

The probability that a randomly selected female aged 60 years old will survive the year is 99.186% according to the National Vital Statistics Report, Vol. 47, No. 28. What is the probability that four randomly selected 60 year old females will survive the year?

3 Compute At-Least Probabilities

EXAMPLE Computing "at least" Probabilities

The probability that a randomly selected female aged 60 years old will survive the year is 99.186% according to the National Vital Statistics Report, Vol. 47, No. 28. What is the probability that at least one of 500 randomly selected 60 year old females will die during the course of the year?

Summary: Rules of Probability

1. The probability of any event must be between 0 and 1, inclusive. If we let E denote any event, then $0 \leq P(E) \leq 1$.
2. The sum of the probabilities of all outcomes in the sample space must equal 1. That is, if the sample space $S = \{e_1, e_2, \ldots, e_n\}$, then $P(e_1) + P(e_2) + \cdots + P(e_n) = 1$.
3. If E and F are disjoint events, then $P(E \text{ or } F) = P(E) + P(F)$. If E and F are not disjoint events, then $P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$.
4. If E represents any event and E^c represents the complement of E, then $P(E^c) = 1 - P(E)$.
5. If E and F are independent events, then $P(E \text{ and } F) = P(E) \cdot P(F)$.

We will not be covering Section 5.4.

Section 5.5 Counting Techniques

Objectives

1. Solve counting problems using the Multiplication Rule
2. Solve counting problems using permutations
3. Solve counting problems using combinations

1 Solve Counting Problems Using the Multiplication Rule

Multiplication Rule of Counting

If a task consists of a sequence of choices in which there are p selections for the first choice, q selections for the second choice, r selections for the third choice, and so on, then the task of making these selections can be done in

$$p \cdot q \cdot r \cdot \cdots$$

different ways.

EXAMPLE Counting the Number of Possible Meals

The fixed-price dinner at Mabenka Restaurant provides the following choices:

Appetizer: soup or salad

Entrée: baked chicken, broiled beef patty, baby beef liver, or roast beef au jus

Dessert: ice cream or cheesecake

How many different meals can be ordered?

If $n \geq 0$ is an integer, the **factorial symbol, $n!$**, is defined as follows:

$$n! = n(n-1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

$$0! = 1 \qquad 1! = 1$$

2 Solve Counting Problems Using Permutations

A **permutation** is an ordered arrangement in which *r* objects are chosen from *n* distinct (different) objects so that $r \le n$ and repetition is not allowed. The symbol $_nP_r$ represents the number of permutations of *r* objects selected from *n* objects.

The formula for $_nP_r$ can be written in **factorial notation**:

$$
\begin{aligned}
{}_nP_r &= n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-r+1) \\
&= n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-r+1) \cdot \frac{(n-r) \cdot \cdots \cdot 3 \cdot 2 \cdot 1}{(n-r) \cdot \cdots \cdot 3 \cdot 2 \cdot 1} \\
&= \frac{n!}{(n-r)!}
\end{aligned}
$$

Number of Permutations of *n* Distinct Objects Taken *r* at a Time

The number of arrangements of *r* objects chosen from *n* objects, in which

1. the *n* objects are distinct,
2. repetition of objects is not allowed, and
3. order is important,

is given by the formula

$$ {}_nP_r = \frac{n!}{(n-r)!} \qquad \textbf{(1)} $$

EXAMPLE Betting on the Trifecta

In how many ways can horses in a 10-horse race finish first, second, and third?

3 Solve Counting Problems Using Combinations

A **combination** is a collection, without regard to order, in which *r* objects are chosen from *n* distinct objects with $r \le n$ without repetition. The symbol $_nC_r$ represents the number of combinations of *n* distinct objects taken *r* at a time.

Number of Combinations of n Distinct Objects Taken r at a Time

The number of different arrangements of r objects chosen from n objects, in which

1. the n objects are distinct
2. repetition of objects is not allowed, and
3. order is not important

is given by the formula

$$ {}_nC_r = \frac{n!}{r!(n-r)!} \qquad \textbf{(2)} $$

EXAMPLE Simple Random Samples

How many different simple random samples of size 4 can be obtained from a population whose size is 20?