Chapter 5 Probability

Section 5.1 Probability Rules

Objectives

1. Understand random processes and the Law of Large Numbers
2. Apply the rules of probabilities
3. Compute and interpret probabilities using the empirical method
4. Compute and interpret probabilities using the classical method
5. Recognize and interpret subjective probabilities

The word **random** suggests an unpredictable outcome. Predicting outcomes while facing uncertainty can be challenging. Here is a simple example. Can you predict the outcome of rolling a single 6-sided die for one particular roll? That would be challenging. However, we are able to determine the proportion of times we observe rolling a two over many rolls of the die. The process of rolling a single die many times and recording the outcomes is a *simulation*.

Simulation is a technique used to recreate a random event. The goal in simulations is to measure how often a goal is observed.

Activity Rolling a Die

Go to www.pearsonhighered.com/sullivanstats and select the Dice Rolling applet. Or, create an applet in StatCrunch by selecting Applets > Simulation > Dice Rolling. Let the number of sides by 10 and the number of dice be 1. Click Compute!. Under Event in the applet notice, “For the Sum of 1 rolls”. Use the pull-down menu and select = and enter 4 in the dialogue box. Click on Convergence.

(a) Click 5 runs four times. Notice the applet rolls a single die a total of 20 times. What proportion of the rolls resulted in a “four” being rolled?

(b) Click 5 runs four more times (for a total of 40 rolls). What proportion of the rolls resulted in a “four” being rolled?

(c) Click Reset. Click 1000 runs. What proportion of the rolls results in a "four" being rolled? What happens to the relative frequency with which a four is observed as the number of rolls increases?

(d) Click Reset. Click 1000 runs. What proportion of the rolls results in a "four" being rolled? What happens to the relative frequency with which a four is observed as the number of rolls increases? Is this the same result as you obtained in Part (c)? Is the behavior in the short run (fewer rolls) the same as it was in Part (c)?

(e) Based on the results of Parts (c) and (d), what would you conjecture is the probability of rolling a four with a single 10-sided die?

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A **random process** represents scenarios where the outcome of any particular trial of an experiment is unknown, but the proportion (or relative frequency) a particular outcome is observed approaches a specific value.

Probability is a measure of the likelihood of a random phenomenon or chance behavior occurring. Probability describes the long-term proportion with which a certain **outcome** will occur in situations with short-term uncertainty.

Probability deals with experiments that yield random short-term results or **outcomes**, yet reveal long-term predictability.

The long-term proportion in which a certain outcome is observed is the probability of that outcome.

The Law of Large Numbers

As the number of repetitions of a probability experiment increases, the proportion with which a certain outcome is observed gets closer to the probability of the outcome.

Example 1 Illustrating the Law of Large Numbers

Your daily commute to work has you going through an intersection where the light always seems to be red. You start recording data to determine the likelihood of arriving at the light while it is red. The data contain the day number and whether the light was red, or not, for 200 consecutive days in which you drove to work. In the data set, the column Red shows a series of 0s and 1s. In that column a 0 indicates the light was not red and a 1 indicates that the light was red. The column Aggregate Red represents the cumulative number of times the light was red. Open the data "5_1_Example1" from the SullyStats group.

(a) What proportion of the days was the light red after 15 days?

(b) What proportion of the days was the light red after 30 days?

(c) Were you stopped by the train on the 40th day?

(d) Graph the cumulative proportion of days the light was red against the number of days.

(e) What is the estimate of the probability of being stuck at a red light at this intersection during your commute?

The Law of Large Numbers versus the Nonexistant Law of Averages

The Law of Large Numbers has intuitive feel (that is, seems to be common sense). However, the Law of Large Numbers is often interpreted as a nonexistent law called the *Law of Averages* by folks who misunderstand the Law of Large Numbers. For example, in baseball you may hear an announcer say that a certain player is *due* to get a hit because he has gone a number of at-bats without getting a hit. Or, you may hear a mother of four boys say she is more likely to have a girl now. In both instances, the confusion is between what happens in the long run (the Law of Large Numbers) with what might happen on the next trial of a probability experiment. That is to say, given that you had four boys you are due for a girl with your fifth child? No, the likelihood of a girl is the same on the fifth child as it was for the first child. Think of it this way, the biology of determining the gender of a child does not look back at the first four trials (first four children) and say, "okay, it's time for a girl." Another way to think about this is that the trials are "memoryless". That is to say, the trials don't recall what has happened in the past – they only consider the next trial.

Activity Illustrating the Nonexistent Law of Averages

Assume the likelihood of having a girl equals the likelihood of having a boy, so $P(\text{boy}) = 0.5$.

(a) Simulate 2000 different families where they have four children where 0 represents a girl and 1 represents a boy.

(b) Determine the number of boys in each family.

(c) Now simulate having a fifth child for all 2000 families.

(d) Among those families where the first four children were all boys, what is the likelihood the fifth child is a girl? Is the family "due" to have a girl?

In probability, an **experiment** is any process that can be repeated in which the results are uncertain.

The **sample space**, S, of a probability experiment is the collection of all possible outcomes.

An **event** is any collection of outcomes from a probability experiment. An event consists of one outcome or more than one outcome. We will denote events with one outcome, sometimes called *simple events*, e_i. In general, events are denoted using capital letters such as E.

Example 2 Identifying Events and the Sample Space of a Probability Experiment

(a) Identify the outcomes of the probability experiment.

(b) Determine the sample space.

(c) Define the event E = "have one boy".

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2 Apply the Rules of Probabilities

Rules of Probabilities

1. The probability of any event E, $P(E)$, must be greater than or equal to 0 and less than or equal to 1. That is, $0 \leq P(E) \leq 1$.
2. The sum of the probabilities of all outcomes must equal 1. That is, if the sample space $S = \{e_1, e_2, \ldots, e_n\}$, then

$$P(e_1) + P(e_2) + \cdots + P(e_n) = 1$$

A **probability model** lists the possible outcomes of a probability experiment and each outcome's probability. A probability model must satisfy rules 1 and 2 of the rules of probabilities.

Example 3 A Probability Model

In a bag of peanut M&M milk chocolate candies, the colors of the candies can be brown, yellow, red, blue, orange, or green. Suppose that a candy is randomly selected from a bag. The table shows each color and the probability of drawing that color. Verify this is a probability model.

Color	Probability
Brown	0.12
Yellow	0.15
Red	0.12
Blue	0.23
Orange	0.23
Green	0.15

■

Key Concepts of Probabilities

- If an event is **impossible**, the probability of the event is 0. If an event is a **certainty,** the probability of the event is 1.
- The closer a probability is to 1, the more likely the event will occur.
- The closer a probability is to 0, the less likely the event will occur.
- An event with probability 0.65 is expected to occur about 65 times out of 100.

An **unusual event** is an event that has a low probability of occurring.

3 Compute and Interpret Probabilities Using the Empirical Method

Approximating Probabilities Using the Empirical Approach

The probability of an event E is approximately the number of times event E is observed divided by the number of repetitions of the experiment.

$$P(E) \approx \text{relative frequency of } E = \frac{\text{frequency of } E}{\text{number of trials of experiment}} \qquad \textbf{(1)}$$

Example 4 Build a Probability Model from a Random Process

Pass the Pigs™ is a Milton-Bradley game in which pigs are used as dice. Points are earned based on the way the pig lands. There are six possible outcomes when one pig is tossed. A class of 52 students rolled pigs 3,939 times. The number of times each outcome occurred is recorded in the table.

Outcome	**Frequency**
Side with no dot	1344
Side with dot	1294
Razorback	767
Trotter	365
Snouter	137
Leaning Jowler	32

(a) Use the results of the experiment to build a probability model for the way the pig lands.

(b) Estimate the probability that a thrown pig lands on the "side with dot".

(c) Would it be unusual to throw a "Leaning Jowler"?

4 Compute and Interpret Probabilities Using the Classical Method

The classical method of computing probabilities requires **equally likely outcomes.**

An experiment is said to have **equally likely outcomes** when each simple event has the same probability of occurring.

Computing Probability Using the Classical Method

If an experiment has n equally likely outcomes and if the number of ways that an event E can occur is m, then the probability of E, $P(E)$, is

$$P(E) = \frac{\text{number of ways that } E \text{ can occur}}{\text{number of possible outcomes}} = \frac{m}{n} \tag{2}$$

So, if S is the sample space of this experiment,

$$P(E) = \frac{N(E)}{N(S)} \tag{3}$$

where $N(E)$ is the number of outcomes in E, and $N(S)$ is the number of outcomes in the sample space.

Example Computing Probabilities Using the Classical Method

Suppose a "fun size" bag of M&Ms contains 9 brown candies, 6 yellow candies, 7 red candies, 4 orange candies, 2 blue candies, and 2 green candies. Suppose that a candy is randomly selected.

(a) What is the probability that it is yellow?

(b) What is the probability that it is blue?

(c) Comment on the likelihood of the candy being yellow versus blue.

5 Recognize and Interpret Subjective Probabilities

A **subjective probability** of an outcome is a probability obtained on the basis of personal judgment.

For example, an economist predicting there is a 20% chance of recession next year would be a subjective probability.

EXAMPLE Empirical, Classical, or Subjective Probability

In his fall 1998 article in **Chance Magazine**, ("A Statistician Reads the Sports Pages," pp. 17-21,) Hal Stern investigated the probabilities that a particular horse will win a race. He reports that these probabilities are based on the amount of money bet on each horse. When a probability is given that a particular horse will win a race, is this empirical, classical, or subjective probability?

Subjective because it is based upon people's feelings about which horse will win the race. The probability is not based on a probability experiment or counting equally likely outcomes. ■

5.2 The Addition Rule and Complements

Objectives

1. Use the Addition Rule for Disjoint Events
2. Use the General Addition Rule (we do not cover this objective)
3. Compute the probability of an event using the Complement Rule

Two events are **disjoint** if they have no outcomes in common. Another name for disjoint events is **mutually exclusive** events.

We often draw pictures of events using **Venn diagrams.** These pictures represent events as circles enclosed in a rectangle. The rectangle represents the sample space, and each circle represents an event. For example, suppose we randomly select a chip from a bag where each chip in the bag is labeled 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Let E represent the event "choose a number less than or equal to 2," and let F represent the event "choose a number greater than or equal to 8." These events are disjoint as shown in the figure.

$$P(E) = \frac{N(E)}{N(S)} = \frac{3}{10} = 0.3$$

$$P(F) = \frac{N(F)}{N(S)} = \frac{2}{10} = 0.2$$

$$P(E \text{ or } F) = \frac{N(E \text{ or } F)}{N(S)} = \frac{5}{10} = 0.5$$
$$= P(E) + P(F) = 0.3 + 0.2 = 0.5$$

Addition Rule for Disjoint Events

If E and F are disjoint (or mutually exclusive) events, then

$$P(E \text{ or } F) = P(E) + P(F)$$

The Addition Rule for Disjoint Events can be extended to more than two disjoint events.

In general, if $E, F, G, \ldots$ each have no outcomes in common (they are pairwise disjoint), then

$$P(E \text{ or } F \text{ or } G \text{ or } \ldots) = P(E) + P(F) + P(G) \cdots$$

Example The Addition Rule for Disjoint Events

The probability model shows the distribution of the number of rooms in housing units in the United States.

Number of Rooms in Housing Unit	Probability
One	0.010
Two	0.032
Three	0.093
Four	0.176
Five	0.219
Six	0.189
Seven	0.122
Eight	0.079
Nine or more	0.080

(a) Verify that this is a probability model.

(b) What is the probability a randomly selected housing unit has two or three rooms?

(c) What is the probability a randomly selected housing unit has one or two or three rooms?

3 Compute the Probability of an Event Using the Complement Rule

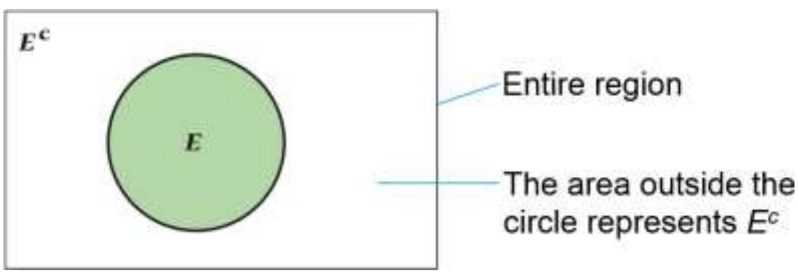
Complement of an Event

Let S denote the sample space of a probability experiment and let E denote an event. The **complement of E**, denoted E^c, is all outcomes in the sample space S that are not outcomes in the event E.

Complement Rule

If E represents any event and E^c represents the complement of E, then

$$P(E^c) = 1 - P(E)$$



EXAMPLE Illustrating the Complement Rule

According to the American Veterinary Medical Association, 31.6% of American households own a dog. What is the probability that a randomly selected household does not own a dog?

Example Computing Probabilities Using Complements

The data to the right represent the travel time to work for residents of Hartford County, CT.

Travel Time	Frequency
Less than 5 minutes	24,358
5 to 9 minutes	39,112
10 to 14 minutes	62,124
15 to 19 minutes	72,854
20 to 24 minutes	74,386
25 to 29 minutes	30,099
30 to 34 minutes	45,043
35 to 39 minutes	11,169
40 to 44 minutes	8,045
45 to 59 minutes	15,650
60 to 89 minutes	5,451
90 or more minutes	4,895

Source: United States Census Bureau

(a) What is the probability a randomly selected resident has a travel time of 90 or more minutes?

(b) Compute the probability that a randomly selected resident of Hartford County, CT will have a commute time less than 90 minutes.

Section 5.3 Independence and the Multiplication Rule

Objectives

1. Identify independent events
2. Use the Multiplication Rule for Independent Events
3. Compute at-least probabilities

1 Identify Independent Events

Two events E and F are **independent** if the occurrence of event E in a probability experiment does not affect the probability of event F. Two events are **dependent** if the occurrence of event E in a probability experiment affects the probability of event F.

EXAMPLE Independent or Not?

(a) Suppose you draw a card from a standard 52-card deck of cards and then roll a die. The events "draw a heart" and "roll an even number" are independent because the results of choosing a card do not impact the results of the die toss.

(b) Suppose two 40-year old women who live in the United States are randomly selected. The events "woman 1 survives the year" and "woman 2 survives the year" are independent.

(c) Suppose two 40-year old women live in the same apartment complex. The events "woman 1 survives the year" and "woman 2 survives the year" are dependent.

Disjoint Events versus Independent Events Disjoint events and independent events are different concepts. Recall that two events are disjoint if they have no outcomes in common, that is, if knowing that one of the events occurs, we know the other event did not occur. Independence means that one event occurring does not affect the probability of the other event occurring. Therefore, knowing two events are disjoint means that the events are not independent.

2 Use the Multiplication Rule for Independent Events

Multiplication Rule for Independent Events

If E and F are independent events, then

$$P(E \text{ and } F) = P(E) \cdot P(F)$$

EXAMPLE Computing Probabilities of Independent Events

The probability that a randomly selected female aged 60 years old will survive the year is 99.186% according to the National Vital Statistics Report, Vol. 47, No. 28. What is the probability that two randomly selected 60 year old females will survive the year?

EXAMPLE Computing Probabilities of Independent Events

A manufacturer of exercise equipment knows that 10% of their products are defective. They also know that only 30% of their customers will actually use the equipment in the first year after it is purchased. If there is a one-year warranty on the equipment, what proportion of the customers will actually make a valid warranty claim?

Multiplication Rule for *n* Independent Events

If events $E_1, E_2, E_3, \ldots, E_n$ are independent, then

$$P(E_1 \text{ and } E_2 \text{ and } E_3 \text{ and } \ldots \text{ and } E_n) = P(E_1) \cdot P(E_2) \cdot \cdots \cdot P(E_n)$$

EXAMPLE Illustrating the Multiplication Principle for Independent Events

The probability that a randomly selected female aged 60 years old will survive the year is 99.186% according to the National Vital Statistics Report, Vol. 47, No. 28. What is the probability that four randomly selected 60 year old females will survive the year?

3 Compute At-Least Probabilities

EXAMPLE Computing "at least" Probabilities

The probability that a randomly selected female aged 60 years old will survive the year is 99.186% according to the National Vital Statistics Report, Vol. 47, No. 28. What is the probability that at least one of 500 randomly selected 60 year old females will die during the course of the year?

Summary: Rules of Probability

1. The probability of any event must be between 0 and 1, inclusive. If we let E denote any event, then $0 \le P(E) \le 1$.
2. The sum of the probabilities of all outcomes in the sample space must equal 1. That is, if the sample space $S = \{e_1, e_2, \dots, e_n\}$, then $P(e_1) + P(e_2) + \cdots + P(e_n) = 1$.
3. If E and F are disjoint events, then $P(E \text{ or } F) = P(E) + P(F)$. If E and F are not disjoint events, then $P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$.
4. If E represents any event and E^c represents the complement of E, then $P(E^c) = 1 - P(E)$.
5. If E and F are independent events, then $P(E \text{ and } F) = P(E) \cdot P(F)$.

We will not be covering Section 5.4.

Section 5.5 Counting Techniques

Objectives

1. Solve counting problems using the Multiplication Rule
2. Solve counting problems using permutations
3. Solve counting problems using combinations

1 Solve Counting Problems Using the Multiplication Rule

Multiplication Rule of Counting

If a task consists of a sequence of choices in which there are p selections for the first choice, q selections for the second choice, r selections for the third choice, and so on, then the task of making these selections can be done in

$$p \cdot q \cdot r \cdot \cdots$$

different ways.

EXAMPLE Counting the Number of Possible Meals

The fixed-price dinner at Mabenka Restaurant provides the following choices:

Appetizer: soup or salad

Entrée: baked chicken, broiled beef patty, baby beef liver, or roast beef au jus

Dessert: ice cream or cheesecake

How many different meals can be ordered?

If $n \geq 0$ is an integer, the **factorial symbol, $n!$**, is defined as follows:

$$n! = n(n-1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

$$0! = 1 \qquad 1! = 1$$

2 Solve Counting Problems Using Permutations

A **permutation** is an ordered arrangement in which r objects are chosen from n distinct (different) objects so that $r \leq n$ and repetition is not allowed. The symbol ${}_nP_r$ represents the number of permutations of r objects selected from n objects.

The formula for ${}_nP_r$ can be written in **factorial notation**:

$$\begin{aligned} {}_nP_r &= n\cdot(n-1)\cdot(n-2)\cdot\cdots\cdot(n-r+1) \\ &= n\cdot(n-1)\cdot(n-2)\cdot\cdots\cdot(n-r+1)\cdot\frac{(n-r)\cdot\cdots\cdot 3\cdot 2\cdot 1}{(n-r)\cdot\cdots\cdot 3\cdot 2\cdot 1} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

Number of Permutations of *n* Distinct Objects Taken *r* at a Time

The number of arrangements of r objects chosen from n objects, in which

1. the n objects are distinct,
2. repetition of objects is not allowed, and
3. order is important,

is given by the formula

$$ {}_nP_r = \frac{n!}{(n-r)!} \quad \textbf{(1)} $$

EXAMPLE Betting on the Trifecta

In how many ways can horses in a 10-horse race finish first, second, and third?

3 Solve Counting Problems Using Combinations

A **combination** is a collection, without regard to order, in which r objects are chosen from n distinct objects with $r \leq n$ without repetition. The symbol ${}_nC_r$ represents the number of combinations of n distinct objects taken r at a time.

Number of Combinations of n Distinct Objects Taken r at a Time

The number of different arrangements of r objects chosen from n objects, in which

1. the n objects are distinct
2. repetition of objects is not allowed, and
3. order is not important

is given by the formula

$$ {}_nC_r = \frac{n!}{r!(n-r)!} \tag{2} $$

EXAMPLE Simple Random Samples

How many different simple random samples of size 4 can be obtained from a population whose size is 20?

5.6 Simulation

Objective

① Use Simulation to Obtain Probabilities

We introduced the idea of simulation in Section 5.1 where we used a computer to randomly roll a die many times. We called this form of simulation a *random process* because the outcome of any particular roll of the die could not be determined ahead of time, but over the course of many rolls, the proportion of time a four was observed settled down to a specific value – namely, 0.17. We called this proportion the probability of rolling a four. In doing these simulations, we relied on statistical applets to randomly generate outcomes using coins or dice to represent results from a random event.

Now we look another form of simulation. In Chapter 1 we discussed the role that randomness plays in data collection. When collecting data for an observational study, it is important that individuals are *randomly selected* to be in the study. This allows the results of the study to be extended to the population from which the individuals were randomly selected. When collecting data for a designed experiment, it is important that the individuals are *randomly assigned* to the various treatment groups in the study. This allows us to make statements of causation between the levels of treatment and the response variable in the study.   In the next example, we look at the role of randomness in randomly selecting individuals to be in a study. We will consider random assignment later in the course.

The randomness in selecting or assigning individuals leads to outcomes that are uncertain. For example, polling agencies (such as Gallup) routinely randomly select about 1000 individuals and pose a question to them such as “Do you believe the amount of income tax you pay to the federal government is fair?” Because the individuals are randomly selected, the number of individuals out of 1000 who believe the amount of federal income tax they pay is fair will likely be different for two different repetitions of the study. Simulation can be used to help describe these differences.

EXAMPLE   Simulating a Poll

In a recent Pew Research poll, it was reported that 43% of adult Americans (aged 18 and older) believe the reason a person is rich is because the individual “had advantages in life.”   There are approximately 241,000,000 adult Americans in the United States.

(a) Simulate obtaining a simple random sample of size 500 from the population. How many of the individuals sampled believe a person is rich because the individual had advantages in life? How many do not believe a person is rich because the individual had advantages in life? What proportion believe a person is rich because the individual had advantages in life?

(b) Simulate obtaining a second simple random sample of size 500 from the population. How many of the individuals sampled believe a person is rich because the individual had advantages in life? How many do not believe a person is rich because the individual had advantages in life? What proportion believe a person is rich because the individual had advantages in life?

(c) Now simulate obtaining at least 2000 more simple random samples of size 500 from the population. Based on the simulation, what is the probability of obtaining a random sample where the proportion who believe a person is rich because the individual had advantages in life is greater than 0.50? Would it be unusual to obtain a sample proportion greater than 0.5 from this population? Explain.