Chapter 6 Discrete Probability Distributions

Objectives
❶ Distinguish between discrete and continuous random variables
❷ Identify discrete probability distributions
❸ Graph discrete probability distributions
❹ Compute and interpret the mean of a discrete random variable
❺ Interpret the mean of a discrete random variable as an expected value
❻ Compute the standard deviation of a discrete random variable

❶ Distinguish between Discrete and Continuous Random Variables

A **random variable** is a numerical measure of the outcome of a probability experiment, so its value is determined by chance. Random variables are typically denoted using capital letters such as X.

A **discrete random variable** has either a finite or countable number of values. The values of a discrete random variable can be plotted on a number line with space between each point. See Figure 1(a).

A **continuous random variable** has infinitely many values. The values of a continuous random variable can be plotted on a line in an uninterrupted fashion. See Figure 1(b).

Figure 1

(a) Discrete Random Variable **(b)** Continuous Random Variable

Example Distinguishing between Discrete and Continuous Random Variables

A group of students plan to purchase a 16 ounce bag of M&M's and record data on the contents. For each of the following variables, state whether it is discrete or continuous. State possible values for the random variable.

a. Mass of each of the M&M's

b. Number of pieces of candy in the bag

c. Diameter of each piece of candy

d. Number of distinct colors of M&M's in the bag

e. The cost to purchase the bag of M&M's

❷ Identify Discrete Probability Distributions

The **probability distribution** of a discrete random variable X provides the possible values of the random variable and their corresponding probabilities. A probability distribution can be in the form of a table, graph, or mathematical formula.

Rules for a Discrete Probability Distribution

Let $P(x)$ denote the probability that the random variable X equals x; then

1. $\sum P(x) = 1$

2. $0 \leq P(x) \leq 1$

Example A Discrete Probability Distribution

Daniel Reisman, of Niverville, New York, submitted the following question to Marilyn vos Savant's December 27, 1998, Parade Magazine column, "Ask Marilyn:"

> At a monthly 'casino night,' there is a game called Chuck-a-Luck: Three dice are rolled in a wire cage. You place a bet on any number from 1 to 6. If any one of the three dice comes up with your number, you win the amount of your bet. (You also get your original stake back.) If more than one die comes up with your number, you win the amount of your bet for each match. For example, if you had a $1 bet on number 5, and each of the dice came up with 5, you would win $3. It appears that the odds of winning are 1 in 6 for each of the three dice, for a total of 3 out of 6 - or 50%. Adding the possibility of having more than one die come up with your number, the odds would seem to be in the gambler's favor. What are the odds of winning this game? I can't believe that a casino game would favor the gambler.

Daniel computed the probabilities incorrectly. There are four possible outcomes. (The selected number can match 0, 1, 2, or 3 of the dice.) The random variable X represents the profit from a $1 bet in Chuck-A-Luck. The table below summarizes the probabilities of earning a profit of x dollars from a $1 bet. Verify that this is a discrete probability distribution.

Number of Dice Matching the Chosen Number	Profit	Probability
0	-$1	0.5787
1	$1	0.3472
2	$2	0.0695
3	$3	0.0046

❸ Graph Discrete Probability Distributions

Example   Graphing a Discrete Probability Distribution

Graph the discrete probability distribution from the previous example.

❹ Compute and Interpret the Mean of a Discrete Random Variable

The Mean of a Discrete Random Variable

The mean of a discrete random variable is given by the formula

$$\mu_X = \sum [x \cdot P(x)] \quad \textbf{(1)}$$

where x is the value of the random variable and $P(x)$ is the probability of observing the value x.

Example Computing the Mean of a Discrete Random Variable

The random variable X represents the profit from a $1 bet in Chuck-A-Luck. The table below summarizes the probabilities of earning a profit of x dollars from a $1 bet. Compute and interpret the mean of the random variable X.

Number of Dice Matching the Chosen Number	Profit	Probability
0	-$1	0.5787
1	$1	0.3472
2	$2	0.0695
3	$3	0.0046

How to Interpret the Mean of a Discrete Random Variable

Interpretation of the Mean of a Discrete Random Variable

Suppose an experiment is repeated n independent times and the value of the random variable X is recorded. As the number of repetitions of the experiment increases, the mean value of the n trials will approach μ_X, the mean of the random variable X. In other words, let x_1 be the value of the random variable X after the first experiment, x_2 be the value of the random variable X after the second experiment, and so on. Then

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

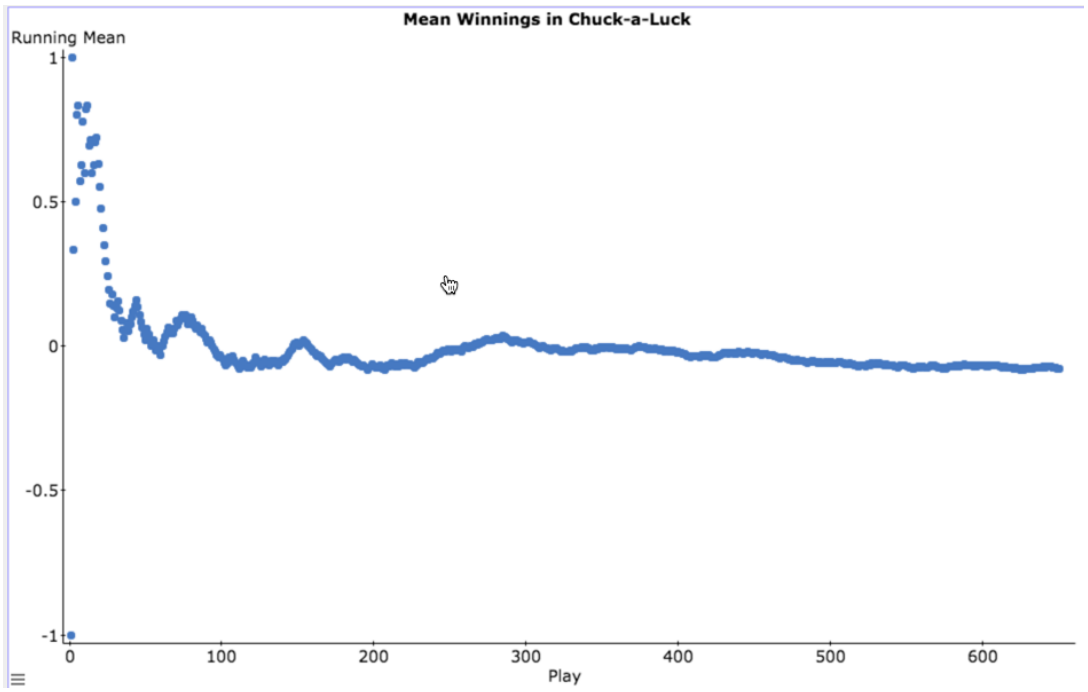
The difference between $\bar{x}$ and μ_X gets closer to 0 as n increases.

Activity – Simulating Chuck-a-Luck

Go to the SullyStats page in StatCrunch and download the "Simulated Chuck-a-Luck Game" data.

Play	Die1	Die2	Die3	Guess	var6	Total Matche	Payout	Cumulative :	Running Mea
1	2	4	1	3		0	-1	-1	-1
2	2	2	2	2		3	3	2	1
3	2	1	5	3		0	-1	1	0.33333333
4	2	3	6	2		1	1	2	0.5
5	2	4	4	4		2	2	4	0.8
6	1	1	5	5		1	1	5	0.83333333
7	4	5	4	1		0	-1	4	0.57142857
8	1	5	3	3		1	1	5	0.625
9	3	1	1	1		2	2	7	0.77777778
10	3	3	6	4		0	-1	6	0.6
11	2	2	2	2		3	3	9	0.81818182
12	4	5	5	4		1	1	10	0.83333333
13	2	6	1	5		0	-1	9	0.69230769
14	6	4	5	5		1	1	10	0.71428571
15	4	3	2	1		0	-1	9	0.6
16	4	4	6	6		1	1	10	0.625
17	3	3	4	3		2	2	12	0.70588235
18	5	6	3	5		1	1	13	0.72222222
19	2	4	5	3		0	-1	12	0.63157895
20	3	2	3	1		0	-1	11	0.55
21	5	1	1	6		0	-1	10	0.47619048
22	4	3	4	2		0	-1	9	0.40909091
23	1	1	2	4		0	-1	8	0.34782609
24	4	5	2	1		0	-1	7	0.29166667
25	1	5	4	2		0	-1	6	0.24
26	2	1	1	3		0	-1	5	0.19230769
27	4	5	1	6		0	-1	4	0.14814815
28	6	4	2	2		1	1	5	0.17857143
29	5	5	2	6		0	-1	4	0.13793103
30	3	4	5	1		0	-1	3	0.1
31	6	2	6	2		1	1	4	0.12903226
32	5	4	2	5		1	1	5	0.15625
33	6	6	3	1		0	-1	4	0.12121212
34	5	4	5	6		0	-1	3	0.08823529
35	1	4	5	2		0	-1	2	0.05714285
36	2	2	4	6		0	-1	1	0.02777777
37	5	1	4	1		1	1	2	0.05405405
38	3	5	4	3		1	1	3	0.07894736
39	4	2	5	6		0	-1	2	0.05128205
40	3	1	5	5		1	1	3	0.075
41	6	6	4	4		1	1	4	0.09756097
42	6	1	4	6		1	1	5	0.11904762
43	1	4	6	4		1	1	6	0.13953488
44	6	4	4	6		1	1	7	0.15909091
45	5	2	6	3		0	-1	6	0.13333333
46	3	6	2	1		0	-1	5	0.10869565
47	1	4	5	3		0	-1	4	0.08510638
48	1	3	3	6		0	-1	3	0.0625
49	6	1	5	4		0	-1	2	0.04081632
50	5	1	1	6		0	-1	1	0.02
51	3	3	1	3		2	2	3	0.05882352
52	5	6	2	4		0	-1	2	0.03846153
53	2	3	4	1		0	-1	1	0.01886792
54	3	2	6	5		0	-1	0	0
55	2	6	5	6		1	1	1	0.01818181
56	6	5	6	2		0	-1	0	0
57	6	3	5	4		0	-1	-1	-0.0175438

The columns Die1,Die2,Die3 represent the outcomes of the three die. The column Guess represents the player's guess on the number. The column Payout represents the payout for the game. The column "Running Mean" is the cumulative payout divided by the number of games played. Notice in the graph below the player starts out "hot" but cools off where the mean winnings approaches the theoretical mean of -\$0.079. In fact, after 650 simulated games, the mean winnings is \$-0.08! See the actual results of the simulation by searching for "Simulated Chuck-a-Luck Game" in StatCrunch.



❺ Interpret the Mean of a Discrete Random Variable as an Expected Value

Because the mean of a random variable represents what we would expect to happen in the long run, it is also called the expected value, $E(X)$. The interpretation of expected value is the same as the interpretation of the mean of a discrete random variable.

Example Finding the Expected Value

Shawn and Maddie purchase a foreclosed property for \$50,000 and spend an additional \$27,000 fixing up the property. They feel that they can resell the property for \$120,000 with probability 0.15, \$100,000 with probability 0.45, \$80,000 with probability 0.25, and \$60,000 with probability 0.15. Compute and interpret the expected profit for reselling the property.

6 Compute the Standard Deviation of a Discrete Random Variable

Standard Deviation of a Discrete Random Variable

The standard deviation of a discrete random variable X is given by

$$\sigma_X = \sqrt{\Sigma[(x - \mu_X)^2 \cdot P(x)]} \quad \textbf{(2a)}$$

$$= \sqrt{\Sigma[x^2 \cdot P(x)] - \mu_X^2} \quad \textbf{(2b)}$$

where x is the value of the random variable, μ_X is the mean of the random variable, and $P(x)$ is the probability of observing a value of the random variable.

Example Finding the Standard Deviation of a Discrete Random Variable

The random variable X represents the profit from a $1 bet in Chuck-A-Luck. The table below summarizes the probabilities of earning a profit of x dollars from a $1 bet. Compute the standard deviation of the random variable X.

Number of Dice Matching the Chosen Number	Profit	Probability
0	-$1	0.5787
1	$1	0.3472
2	$2	0.0695
3	$3	0.0046

6.2 The Binomial Probability Distribution

Objectives

❶ Determine whether a probability experiment is a binomial experiment
❷ Compute probabilities of binomial experiments
❸ Compute the mean and standard deviation of a binomial random variable
❹ Graph a binomial probability distribution

❶ Determine Whether a Probability Experiment Is a Binomial Experiment

Criteria for a Binomial Probability Experiment

An experiment is said to be a **binomial experiment** if

1. The experiment is performed a fixed number of times. Each repetition of the experiment is called a **trial**.
2. The trials are independent. This means that the outcome of one trial will not affect the outcome of the other trials.
3. For each trial, there are two mutually exclusive (disjoint) outcomes: success or failure.
4. The probability of success is the same for each trial of the experiment.

Criteria for a Binomial Probability Experiment

An experiment is said to be a **binomial experiment** if

1. The experiment is performed a fixed number of times. Each repetition of the experiment is called a **trial**.
2. The trials are independent. This means that the outcome of one trial will not affect the outcome of the other trials.
3. For each trial, there are two mutually exclusive (disjoint) outcomes: success or failure.
4. The probability of success is the same for each trial of the experiment.

Example Identifying Binomial Experiments

Which of the following is a binomial experiment? For any binomial experiment, identify the random variable X, the value of p, and the potential number of successes, x.

(a) A player rolls a pair of fair die 10 times. The number X of 7's rolled is recorded.

(b) The proportion of flights that arrived on-time according to the U.S. Department of Transportation in July, 2016 was 0.7515. In order to assess reasons for delays, an official with the FAA randomly selects flights from July, 2016 until she finds 10 that were not on time. The number of flights *X* that need to be selected is recorded.

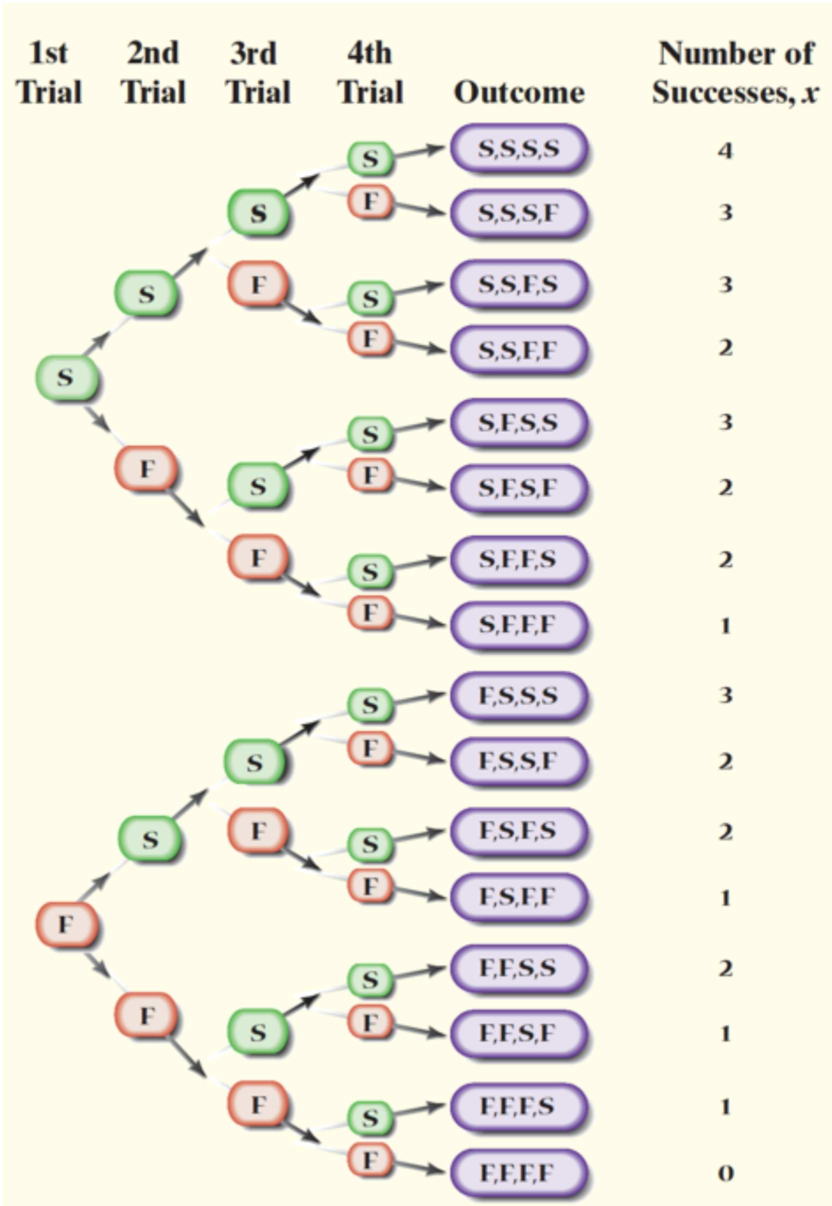
(c) In a class of 30 students, 55% are female. The instructor randomly selects 4 students. The number *X* of females selected is recorded.

❷ Compute Probabilities of Binomial Experiments

EXAMPLE Constructing a Binomial Probability Distribution

According to the U.S. Department of Transportation, the proportion of flights that arrive on time was 0.70 in July, 2017 for all flights arriving at Chicago's O'Hare International Airport. Suppose that 4 flights are randomly selected from July, 2017 and the number of on time flights *X* is recorded.

Construct a probability distribution for the random variable *X* using a tree diagram.


1st Trial
2nd Trial
3rd Trial
4th Trial
Outcome
Number of Successes, x
S,S,S,S 4
S,S,S,F 3
S,S,F,S 3
S,S,F,F 2
S,F,S,S 3
S,F,S,F 2
S,F,F,S 2
S,F,F,F 1
F,S,S,S 3
F,S,S,F 2
F,S,F,S 2
F,S,F,F 1
F,F,S,S 2
F,F,S,F 1
F,F,F,S 1
F,F,F,F 0

Binomial Probability Distribution Function

The probability of obtaining x successes in n independent trials of a binomial experiment is given by

$$P(x) = {}_{n}C_{x}\, p^{x}(1-p)^{n-x} \qquad x = 0, 1, 2, \ldots, n \tag{1}$$

where p is the probability of success.

Phrase	**Math Symbol**
"at least" or "no less than" or "greater than or equal to"	$\geq$
"more than" or "greater than"	$>$
"fewer than" or "less than"	$<$
"no more than" or "at most" or "less than or equal to"	$\leq$
"exactly" or "equals" or "is"	$=$

Example Computing Binomial Probabilities

According to Harris International, 58% of all adult Americans believe that divorce is acceptable.   In a random sample of 30 adult Americans, what is the probability that
(a) exactly 20 believe that divorce is acceptable?
(b) fewer than 10 believe that divorce is acceptable?
(c) at least 10 believe that divorce is acceptable?
(d) the number who believe divorce is acceptable is between 18 and 22, inclusive?

❸ Compute the Mean and Standard Deviation of a Binomial Random Variable

Mean (or Expected Value) and Standard Deviation of a Binomial Random Variable

A binomial experiment with n independent trials and probability of success p has a mean and standard deviation given by the formulas

$$\mu_X = np \quad \text{and} \quad \sigma_X = \sqrt{np(1-p)} \qquad \textbf{(2)}$$

Example Finding the Mean and Standard Deviation of a Binomial Random Variable

According to Harris International, 58% of all adult Americans believe that divorce is acceptable. In a random sample of 500 adult Americans, determine the mean and standard deviation number of adult Americans who believe divorce is acceptable.

❹ Graph a Binomial Probability Distribution

Activity Exploring the Graphs of Binomial Probability Distributions

Go to www.statcrunch.com. Select Applets > Distribution demos. Select the Binomial radio button and click Compute!.

(a) Move the "n" slider to n = 20. Grab the "p" slider to see the role that p plays in the shape of the distribution.

(b) Move the "p" slide to p = 0.2. Let n increase from 10 to 60. Describe what happens to the shape of the distribution as the number of trials, *n*, increases.

For a fixed p, as the number of trials n in a binomial experiment increases, the probability distribution of the random variable X becomes bell shaped. As a rule of thumb, if $np(1 - p) \geq 10$,* the probability distribution will be approximately bell shaped.

Example Using the Mean, Standard Deviation, and Empirical Rule to Check for Unusual Results in a Binomial Experiment

According to Harris International, 58% of all adult Americans believe that divorce is acceptable. Suppose a random sample of 500 adult Americans results in 310 who believe divorce is acceptable. Is this result unusual?